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A SIMPLE C^0 QUADRILATERAL THICK/THIN SHELL ELEMENT BASED ON THE REFINED SHELL THEORY AND THE ASSUMED STRAIN FIELDS

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Abstract—This is a companion paper of the authors' previous one. In this paper, a convenient form of the refined two-dimensional shell theory for the thick cylindrical shells proposed by the authors is given; and a simple and efficient C^0 quadrilateral shell element is developed by the quasi-conforming element technique. The element stiffness matrix presented here is given explicitly. This C^0 thick/thin shell element satisfies the rigid body motion, passes the patch test and exhibits neither shear locking and membrane locking nor spurious kinematic modes. The numerical examples solved here demonstrate that the C^0 quasi-conforming shell element gives very good results in the analysis of both thick and thin shells.

INTRODUCTION

C^0 shell finite elements are simple and efficient elements in the analysis of thin structures. However, many C^0 elements exhibit some deficiencies in the analysis of thin plates and shells, such as the shear locking and spurious kinematic modes. In the past few years, the shear locking and spurious modes of C^0 elements have received increasing attention from researchers (Hughes and Hinton, 1986; Atluri and Yagawa, 1988), and many approaches were proposed to construct reliable and accurate C^0 elements for the analysis of both thick and thin plates and shells. Up to now, most of the C^0 elements are based on the degenerated shell element given by Ahmad *et al.* (1970). The transverse shear locking problem in the degenerated shell element can be solved by the use of the reduced or selective integration technique (Zienkiewicz *et al.*, 1971; Hughes, 1987). However, the reduced integration may result in the development of spurious zero strain energy modes. Consequently, these kind of elements are not very reliable. The employment of discrete Kirchhoff constraints (Wempner *et al.*, 1968; Li *et al.*, 1985) is another approach to avoid shear locking for C^0 elements. Unfortunately, the discrete Kirchhoff constraints lead to complex inversion and calculations in the formulation of the element stiffness matrix (Li *et al.*, 1985). A new method to construct C^0 elements, is to employ the so-called enhanced interpolations of the transverse shear strain and membrane strain (Huang and Hinton, 1986) or assumed natural-coordinate strains (Park and Stanley, 1986). The C^0 elements based on the enhanced strains can overcome the shear locking and spurious kinematic mode problems and give quite good results. However, like all other degenerated elements, numerical integration is used in the formulation of these elements even in the case of flat plate elements. As it is well known, the numerical integration is very time consuming especially in nonlinear problems where the stiffness matrix has to be evaluated many times during the analysis.

It seems that Ashwell (1976) first used the term of strain element. But in his strain elements, the strain functions are only used to form curved finite element shape functions in order to satisfy the rigid body motion of a curved element. Therefore, these strain elements still belong to the conventional assumed displacement elements. In Huang and Hinton's element (1986), only the transverse shear strain and membrane strain are interpolated. In Park and Stanley's element (1986), the strains are interpolated along the so-called reference lines. Consequently, elements given by Huang and Hinton (1986), and Park and Stanley (1986) are not based on the general strain fields.

Tang *et al.* (1980) and Chen and Liu (1980) proposed a simple and fundamental finite element approach called "Quasi-Conforming Element Technique" (QCT). In the quasi-conforming element, the element strain fields are interpolated directly rather than obtained from the assumed displacement field. The element strains can be expressed in terms of the element nodal displacement vector by integration along the element boundaries together with the string net functions, which are similar to the edge displacement interpolations in Pian's (1964) hybrid stress element. Based on the element strain field, the element stiffness matrix can be evaluated in the usual way. Another very important feature of quasi-conforming elements is that all the integrations can be done directly without the employment of the numerical integration. Consequently, the quasi-conforming element technique can give the explicit form of the stiffness matrix. Furthermore, because the strain fields are interpolated directly rather than being derived from the assumed displacement field, the quasi-conforming elements give very accurate stresses (Tang *et al.*, 1980; Shi, 1980). The quasi-conforming element technique is a general finite element method which treats the conforming, non-conforming and hybrid elements in a simple and unified way. Many excellent quasi-conforming elements were obtained for plane stress/strain, plate bending and shell problems (Tang *et al.*, 1980; Shi, 1980; Jiang, 1980; Lu and Liu, 1981; Jin, 1980). A brief introduction for these elements is given by Tang *et al.* (1983). Several unified C^1 thin/thick beam and plate elements were presented by Liu *et al.* (1984).

There are two objectives in this paper. The first is to give a convenient form for the application of the refined two-dimensional theory of the thick cylindrical shells proposed by the authors (Voyiadjis and Shi, 1990). The second is to develop an efficient and accurate C^0 thick/thin shell element based on the refined shell theory and the quasi-conforming element technique.

There are many shell theories in the literature. All the theories could give good results in the analysis of thin shells. Nevertheless, only a few theories can account for the transverse shear deformation, the initial curvature effect, and the contribution of the transverse normal stress in the analysis of thick shells. A refined two-dimensional theory for thick cylindrical shells was proposed by the authors (Voyiadjis and Shi, 1990). This refined shell theory does not only incorporate the effect of the transverse shear deformation, but also takes account of the effect of the initial curvature and the radial stress. Consequently, the proposed refined theory gives a very good approximation for the shell constitutive equations. The refined shell theory was applied to the analysis of thick circular arches in the authors' previous paper. The numerical examples given by Voyiadjis and Shi (1990) indicate that this theory can give very good results even for extremely thick arches where the ratio of radius to thickness R/h is 3. However, the stress resultants and stress couples in the proposed refined shell theory are not symmetric due to the incorporation of the initial curvature effect. These unsymmetric stress resultants and couples are not convenient for use in the finite element analysis. The effective stress resultants and couples are used in this paper to make the stress resultant and couple tensors symmetric. The strain components which correspond to the effective stress resultants and couples are also given here. Based on the developed refined shell theory, a coupled strain energy density is proposed which provides the foundation for the C^0 assumed strain element developed in this paper.

The rigid body motion is a difficult problem to be satisfied in the displacement based curved finite elements. But the rigid body motion can be satisfied automatically in the quasi-conforming elements by the assumed strain fields. The spurious mechanism which is a serious problem in the reduced integration elements can be avoided in the quasi-conforming element by the properly chosen strain fields. The rank of the element stiffness matrix of the quasi-conforming elements can be checked *a priori* by the given element nodal displacement vector, the rigid body modes, the assumed strain fields, and the compatibility equations of the displacement fields. A general approach for the rank analysis of the element stiffness matrix was given by Liu *et al.* (1983). The compatibility equations of the displacements can be satisfied *a priori* in the assumed strain fields. Nevertheless, it will result in more calculations in the formulation of the element stiffness matrix without much improvement of the accuracy (Shi, 1980). Therefore, the compatibility equations are only used in the rank analysis but are not enforced in the assumed strain fields. A good C^0 thick/thin shell element should satisfy the Kirchhoff–Love hypothesis in the case of the thin plates or shells. The present C^0 thick/thin shell element satisfies the Kirchhoff–Love hypothesis by the simple dependent displacement and rotation interpolations of a straight beam. Therefore, the C^0 quasi-conforming element presented in this paper exhibits neither shear and membrane locking nor spurious kinematic mode in both thick and thin shell analyses. This C^0 element does not only pass the patch test, but also gives very good results for both thick and thin shells. Two numerical examples are given to illustrate the behavior of the C^0 quasi-conforming shell element.

EFFECTIVE STRESS RESULTANTS, STRESS COUPLES AND THE CORRESPONDING STRAIN COMPONENTS

In the refined two-dimensional theory of thick cylindrical shells proposed by Voyiadjis and Shi (1990), the average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ along the normal at a point on the middle surface and the average rotations ϕ_x, ϕ_y of the normal are used in the analysis. These variables are employed instead of the widely used displacements u_0, v_0 and w_0 on the middle surface of the shell. The average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ and the average rotations ϕ_x, ϕ_y are defined as

$$\bar{u} = u_0 - \frac{h^2}{24Er^2}\left[\frac{1}{c_1}\frac{\partial p_i}{\partial x}(R^2 - r_2^2) + \frac{1}{c_2}\frac{\partial p_0}{\partial x}(R^2 - r_1^2)\right] \tag{1}$$

$$\bar{v} = v_0 - \frac{h^2}{24Er^2}\left[\frac{1}{c_1}\frac{\partial p_i}{\partial x}(R^2 - r_2^2) + \frac{1}{c_2}\frac{\partial p_0}{\partial y}(R^2 - r_1^2)\right] \tag{2}$$

$$\bar{w} = w_0 - \frac{3v}{10hE}(M_x + M_y) + \frac{h^2}{20ER^3}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_0}{c_2}r_1^2\right) \tag{3}$$

$$\phi_x = \frac{\partial \bar{w}}{\partial x} - \gamma_x \tag{4}$$

and

$$\phi_y = \frac{\partial \bar{w}}{\partial y} - \gamma_y - \frac{\bar{v}}{R}. \tag{5}$$

In the above equations, h is the thickness of the shell, E is Young's modulus, v is Poisson's ratio, R is the radius of the middle surface, p_0 and p_i are the normal pressures on the outside and inside surfaces of the shell, respectively, as shown in Fig. 1, and γ_x and γ_y are the

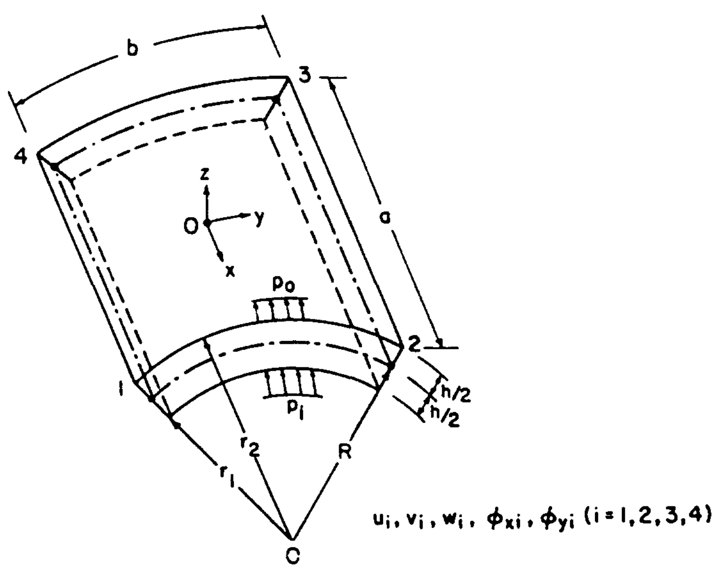

Fig. 1. Thick shell element.

transverse shear deformations in the generator and circumferential directions, respectively. Referring to eqns (1)–(5), r_1, r_2, c_1 and c_2 are, respectively, expressed as follows:

$$r_1 = R - \frac{h}{2} \tag{6a}$$

$$r_2 = R + \frac{h}{2} \tag{6b}$$

$$c_1 = \left(\frac{r_2}{r_1}\right)^2 - 1 \tag{6c}$$

$$c_2 = 1 - \left(\frac{r_1}{r_2}\right)^2. \tag{6d}$$

The modifying terms in eqns (1)–(3) have resulted from the radial stress component σ_z.

The constitutive equations for this shell theory, in the case of isotropic linear-elastic material, are expressed in terms of $\bar{u}$, $\bar{v}$, $\bar{w}$, ϕ_x and ϕ_y as follows:

$$M_x = D\left[\frac{\partial \phi_x'}{\partial x} + v\frac{\partial \phi_\phi}{\partial y}\right] + k_1 p_i + k_2 p_0 \tag{7}$$

$$M_y = D\left[\frac{\partial \phi_y}{\partial y} + v\frac{\partial \phi_x'}{\partial x} + \frac{1}{R}\left(\frac{\bar{w}}{R} + v\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)\right] + k_3 p_i + k_4 p_0 \tag{8}$$

$$M_{xy} = D\left(\frac{1-v}{2}\right)\left[\frac{\partial \phi_x'}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{1}{R}\left(\frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x}\right)\right] \tag{9a}$$

$$M_{yx} = D\left(\frac{1-v}{2}\right)\left[\frac{\partial \phi_x'}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{2}{R}\frac{\partial \bar{u}}{\partial y}\right] \tag{9b}$$

$$N_x = S\left(\frac{\partial \bar{u}}{\partial x} + v\frac{\partial \bar{v}}{\partial y} + v\frac{\bar{w}}{R}\right) - \frac{D}{R}\left(\frac{\partial \phi_x'}{\partial x} + \frac{1}{R}\frac{\partial \bar{u}}{\partial x}\right) + k_5 p_i + k_6 p_0 \tag{10}$$

$$N_y = S\left(\frac{\partial \bar{v}}{\partial y} + \frac{\bar{w}}{R} + v\frac{\partial \bar{u}}{\partial x}\right) + \frac{D}{R}\left(\frac{\partial \phi_y}{\partial y} + \frac{1}{R}\frac{\partial \bar{v}}{\partial y} + \frac{\bar{w}}{R^2}\right) + k_5 p_i + k_6 p_0 \tag{11}$$

$$N_{xy} = S\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\left(\frac{1-v}{2}\right) - \frac{D}{R}\left(\frac{1-v}{4}\right)\left[\frac{\partial \phi_x'}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{1}{R}\left(\frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x}\right)\right] \tag{12a}$$

$$N_{yx} = S\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\left(\frac{1-v}{2}\right) + \frac{D}{R}\left(\frac{1-v}{4}\right)\left[\frac{\partial \phi_x'}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{1}{R}\left(3\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\right] \tag{12b}$$

$$Q_x = T\left(\frac{\partial \bar{w}}{\partial x} - \phi_x' - \frac{\bar{u}}{R}\right) = T\gamma_x \tag{13}$$

and

$$Q_y = T\left(\frac{\partial \bar{w}}{\partial y} - \phi_y - \frac{\bar{v}}{R}\right) = T\gamma_y \tag{14}$$

where

$$\phi'_x = \phi_x - \frac{\bar{u}}{R}. \tag{4a}$$

In the above equations, x is the coordinate along a generator, $y = R\phi$ is the arc length along the circumferential direction, D is the flexural rigidity, S is the tensile rigidity and T is the shear rigidity. The rigidities are defined as follows:

$$D = \frac{Eh^3}{12(1-v^2)}, \quad S = \frac{Eh}{1-v^2}, \quad T = \frac{5}{6}\frac{Eh}{2(1+v)}. \tag{15}$$

In eqns (7), (8), (10) and (11), the parameters k_i ($i = 1, 2, \ldots, 6$) are the contributions of the radial stress σ_z and are defined as follows:

$$k_1 = -v\frac{D}{ER^3}\frac{1}{c_1}[(2+v)(R^2-r_2^2)+2(1+v)r_2^2] \tag{16}$$

$$k_2 = -v\frac{D}{ER^3}\frac{1}{c_2}[(2+v)(R^2-r_1^2)+2(1+v)r_1^2] \tag{17}$$

$$k_3 = -v\frac{D}{ER^3}\frac{1}{c_1}[R^2-r_2^2+2(1+v)r_2^2] \tag{18}$$

$$k_4 = -v\frac{D}{ER^3}\frac{1}{c_2}[R^2-r_1^2+2(1+v)r_1^2] \tag{19}$$

$$k_5 = \frac{v}{1-v}\frac{h}{R^2}\frac{1}{c_1}(R^2-r_2^2) \tag{20}$$

and

$$k_6 = \frac{v}{1-v}\frac{h}{R^2}\frac{1}{c_2}(R^2-r_1^2). \tag{21}$$

Equations (7)–(14) will result in the C^0 continuity problem in the finite element analysis.

Due to the presence of the initial curvature effect, the stress resultants and couple tensors are unsymmetric. Consequently, the resulting stiffness matrix in the finite element analysis will not be symmetric. The constitutive equations given by eqns (7)–(12b) are not convenient for use in the finite element analysis unless the stress resultants and couple tensors are modified to become symmetric tensors.

The effective twisting stress couple $\bar{M}_{xy}$ and the effective shear stress resultant $\bar{N}_{xy}$ used by Niordson (1985) are adopted here. Therefore, $\bar{M}_{xy}$ and $\bar{N}_{xy}$ are defined, respectively, as follows:

$$\bar{M}_{xy} = \bar{M}_{yx} = \tfrac{1}{2}(M_{xy}+M_{yx}) = D\left(\frac{1-v}{2}\right)\left[\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+\frac{1}{2R}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)\right] \tag{9}$$

$$\bar{N}_{xy} = \bar{N}_{yx} = \tfrac{1}{2}(N_{xy}+N_{yx}) = \frac{1-v}{2}\left(S+\frac{D}{R^2}\right)\left(\frac{\partial\bar{u}}{\partial y}+\frac{\partial\bar{v}}{\partial x}\right). \tag{12}$$

We note that the unsymmetric parts in eqns (9a), (9b), (12a) and (12b) are the terms associated only with $1/R$ or h/R. We therefore conclude that expressions (9) and (12) are consequently very good approximations for expressions (9a) and (9b), and (12a) and (12b), as long as the shell is not extremely thick.

The membrane strains and the curvatures are defined in terms of $\bar{u}$, $\bar{v}$, $\bar{w}$, ϕ_y, and ϕ_x as follows:

$$\varepsilon_x = \frac{\partial \bar{u}}{\partial x} \tag{22}$$

$$\varepsilon_y = \frac{\partial \bar{v}}{\partial y} + \frac{\bar{w}}{R} \tag{23}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \tag{24}$$

$$\kappa_x = \frac{\partial \phi_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) \tag{25}$$

$$\kappa_y = \frac{\partial \phi_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \bar{w}}{\partial y} - \gamma_y - \frac{\bar{v}}{R}\right) \tag{26}$$

$$\kappa_{xy} = \frac{1}{2}\left(\frac{\partial \phi'_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{1}{4R}\left(\frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x}\right). \tag{27}$$

The stress resultants and couples may now be expressed in terms of the strain components given above as follows:

$$M_x = D\left[\kappa_x + \nu\kappa_y - \frac{1}{R}\varepsilon_x\right] + k_1 p_i + k_2 p_0 \tag{28}$$

$$M_y = D\left[\kappa_y + \nu\kappa_x + \frac{1}{R}\varepsilon_y\right] + k_3 p_i + k_4 p_0 \tag{29}$$

$$\bar{M}_{xy} = D(1-\nu)\kappa_{xy} \tag{30}$$

$$N_x = S[\varepsilon_x + \nu\varepsilon_y] - \frac{D}{R}\kappa_x + k_5 p_i + k_6 p_0 \tag{31}$$

$$N_y = S[\varepsilon_y + \nu\varepsilon_x] + \frac{D}{R}\left[\kappa_y + \frac{1}{R}\varepsilon_y\right] + k_5 p_i + k_6 p_0 \tag{32}$$

and

$$\bar{N}_{xy} = S(1-\nu)\left(1 + \frac{D}{R^2 S}\right)\varepsilon_{xy}. \tag{33}$$

It can easily be seen that the stress couples are associated with the membrane strains, and the stress resultants are coupled with the changes of curvatures. This results in a coupled strain energy density.

COUPLED STRAIN ENERGY DENSITY AND THE ELEMENT STIFFNESS MATRIX

Using the effective stress resultant and couple, the strain energy density U may be expressed as follows:

$$U = \tfrac{1}{2}(M_x\kappa_x + M_y\kappa_y + 2\bar{M}_{xy}\kappa_{xy} + N_x\varepsilon_x + N_y\varepsilon_y + 2\bar{N}_{xy}\varepsilon_{xy} + Q_x\gamma_x + Q_y\gamma_y). \tag{34}$$

Substituting eqns (13), (14) and (22)–(33) into the above expression, we obtain the following expression:

$$U = U_b + U_m + U_s + U_{bm} + U_0 \tag{35}$$

where U_b, U_m, U_s and U_{bm} are, respectively, the quadratic functions of curvatures, membrane strains, transverse shear strain, and the coupled curvatures and membrane strains. However, in eqn (35) U_0 is only the linear function of those strains. If we let

$$\boldsymbol{\varepsilon}_b = \{\kappa_x, \kappa_y, 2\kappa_{xy}\}^T \tag{36}$$

$$\boldsymbol{\varepsilon}_m = \{\varepsilon_x, \varepsilon_y, 2\varepsilon_{xy}\}^T \tag{37}$$

and

$$\boldsymbol{\varepsilon}_s = \{\gamma_x, \gamma_y\}^T \tag{38}$$

then, the strain energy quantities U_b, U_m, U_s and U_{bm} may now be expressed in the matrix form as follows:

$$U_b = \tfrac{1}{2}\boldsymbol{\varepsilon}_b^T D \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \boldsymbol{\varepsilon}_b = \tfrac{1}{2}\boldsymbol{\varepsilon}_b^T \mathbf{D} \boldsymbol{\varepsilon}_b \tag{39}$$

$$U_m = \tfrac{1}{2}\boldsymbol{\varepsilon}_m^T S \begin{bmatrix} 1 & v & 0 \\ v & \left(1 + \frac{D}{SR^2}\right) & 0 \\ 0 & 0 & \frac{1-v}{2}\left(1 + \frac{D}{SR^2}\right) \end{bmatrix} \boldsymbol{\varepsilon}_m = \tfrac{1}{2}\boldsymbol{\varepsilon}_m^T \mathbf{S} \boldsymbol{\varepsilon}_m \tag{40}$$

$$U_s = \tfrac{1}{2}\boldsymbol{\varepsilon}_s^T \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \boldsymbol{\varepsilon}_s = \tfrac{1}{2}\boldsymbol{\varepsilon}_s^T \mathbf{T} \boldsymbol{\varepsilon}_s \tag{41}$$

and

$$U_{bm} = \tfrac{1}{2}\boldsymbol{\varepsilon}_b^T \frac{2D}{R} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\varepsilon}_m = \tfrac{1}{2}\boldsymbol{\varepsilon}_b^T \mathbf{F} \boldsymbol{\varepsilon}_m. \tag{42}$$

The strain energy over the domain Ω of a finite element is given by

$$\Pi_e = \iint_\Omega U \, \mathrm{d}\Omega \tag{43}$$

or

$$\Pi_e = \frac{1}{2}\iint_\Omega [\varepsilon_b^T \bar{\mathbf{D}}\varepsilon_b + \varepsilon_m^T \bar{\mathbf{S}}\varepsilon_m + \varepsilon_s^T \bar{\mathbf{T}}\varepsilon_s + \varepsilon_b^T \mathbf{F}\varepsilon_m + U_0]\,\mathrm{d}\Omega. \tag{44}$$

The strains ε_b, ε_m, and ε_s may be expressed in terms of the nodal displacement vector of the element $\mathbf{q}$ as follows:

$$\varepsilon_b = \mathbf{B}_b\mathbf{q} \tag{45}$$

$$\varepsilon_m = \mathbf{B}_m\mathbf{q} \tag{46}$$

$$\varepsilon_s = \mathbf{B}_s\mathbf{q}. \tag{47}$$

Substituting expressions (45), (46) and (47) into equation (44), we obtain the following expression for Π_e

$$\Pi_e = \tfrac{1}{2}\mathbf{q}^T \iint_\Omega [\mathbf{B}_b^T\bar{\mathbf{D}}\mathbf{B}_b + \mathbf{B}_m^T\bar{\mathbf{S}}\mathbf{B}_m + \mathbf{B}_s^T\bar{\mathbf{T}}\mathbf{B}_s + \mathbf{B}_b^T\mathbf{F}\mathbf{B}_m]\,\mathrm{d}\Omega\mathbf{q} + W_0 \tag{48}$$

or

$$\Pi_e = \tfrac{1}{2}\mathbf{q}^T[\mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s + \mathbf{K}_{bm}]\mathbf{q} + W_0 \tag{49}$$

where $\mathbf{K}_b$, $\mathbf{K}_m$, $\mathbf{K}_s$ and $\mathbf{K}_{bm}$ are the element stiffness matrices related to bending, stretch, transverse shear deformation, and coupling between the bending and stretch, respectively. In eqn (49) W_0 is the strain energy associated with the distributed load normal to the middle surface of the shell explicitly. The respective stiffnesses and W_0 are given as follows:

$$\mathbf{K}_b = \iint_\Omega \mathbf{B}_b^T\bar{\mathbf{D}}\mathbf{B}_b\,\mathrm{d}\Omega \tag{50}$$

$$\mathbf{K}_m = \iint_\Omega \mathbf{B}_m^T\bar{\mathbf{S}}\mathbf{B}_m\,\mathrm{d}\Omega \tag{51}$$

$$\mathbf{K}_s = \iint_\Omega \mathbf{B}_s^T\bar{\mathbf{T}}\mathbf{B}_s\,\mathrm{d}\Omega \tag{52}$$

$$\mathbf{K}_{bm} = \iint_\Omega \mathbf{B}_b^T\mathbf{F}\mathbf{B}_m\,\mathrm{d}\Omega \tag{53}$$

and

$$W_0 = \frac{1}{2}\iint_\Omega U_0\,\mathrm{d}\Omega. \tag{54}$$

According to the variational principle, the element stiffness matrix $\mathbf{K}$ is given as

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s + \tfrac{1}{2}(\mathbf{K}_{bm} + \mathbf{K}_{bm}^T). \tag{55}$$

It should be noticed that U_0 will not contribute to the stiffness matrix but to the external force vector since it is only a linear function of the nodal displacement $\mathbf{q}$. Once the strain fields are given in the form of eqns (45)–(47), the element stiffness matrix can be evaluated easily using eqns (50)–(55).

EVALUATING ELEMENT STRAIN FIELD BY THE QUASI-CONFORMING TECHNIQUE

The four node quadrilateral element will be constructed here as the quadrilateral element is the simplest and most efficient element for cylindrical shell analysis. The nodal variables $\bar{u}_i$, $\bar{v}_i$, $\bar{w}_i$, ϕ_{xi} and ϕ_{yi} will be used at each node i ($i = 1, 2, 3, 4$). We therefore have a C^0 continuity problem and twenty degrees of freedom in each element. The quasi-conforming technique (QCT) proposed by Tang *et al.* (1980) is employed to compute the element stiffness matrix in this work. In the assumed displacement method, the strains are evaluated from the assumed displacement field by differentiation. However, the strain field is interpolated directly in the quasi-conforming elements and the strain field is evaluated by integrations along the element boundaries/over the element domain.

According to the given nodal variables, the compatibility equations of the displacement field, and the requirement for the proper rank of the element stiffness matrix (see Liu *et al.*, 1983), the strain fields are interpolated as follows:

(a) *Linear bending strain field*

$$\boldsymbol{\varepsilon}_b = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + \dfrac{1}{2R}\left(\dfrac{\partial \bar{u}}{\partial y} - \dfrac{\partial \bar{v}}{\partial x}\right) \end{Bmatrix}$$

$$= \begin{bmatrix} 1xyxy & & 0 \\ & 1xyxy & \\ 0 & & 1xy \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{10} \\ \alpha_{11} \end{Bmatrix} = \mathbf{P}_b \boldsymbol{\alpha}_b. \quad (56)$$

(b) *"Constant" stretch strain field*

$$\boldsymbol{\varepsilon}_m = \begin{Bmatrix} \dfrac{\partial \bar{u}}{\partial x} \\ \dfrac{\partial \bar{v}}{\partial y} + \dfrac{\bar{w}}{R} \\ \dfrac{\partial \bar{u}}{\partial y} + \dfrac{\partial \bar{v}}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{16} \end{Bmatrix} = \mathbf{P}_m \boldsymbol{\alpha}_m. \quad (57)$$

(c) *Constant transverse shear strain*

$$\boldsymbol{\varepsilon}_s = \begin{Bmatrix} \dfrac{\partial \bar{w}}{\partial x} - \phi_x \\ \dfrac{\partial \bar{w}}{\partial y} - \dfrac{\bar{v}}{R} - \phi_y \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_{17} \\ \alpha_{18} \end{Bmatrix} = \mathbf{P}_s \boldsymbol{\alpha}_s, \quad (58)$$

where the origin is located at the centroid, and $\alpha_1, \alpha_2, \ldots, \alpha_{18}$, are the undetermined strain parameters. Because of the argument given in the introduction, the compatibility equations of the displacement field are not enforced *a priori* in the above strain interpolations.

Let $\mathbf{P}$ be the trial function for the assumed strain field, i.e. $\boldsymbol{\varepsilon} = \mathbf{P}\boldsymbol{\alpha}$, and $\mathbf{N}$ the corresponding test function. The strain parameter $\boldsymbol{\alpha}$ are determined from the quasi-conforming technique as follows:

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{C}\mathbf{q} \tag{59}$$

where $\mathbf{q}$ is the element nodal displacement vector and

$$\mathbf{A} = \iint_{\Omega} \mathbf{N}^T\mathbf{P}\, \mathrm{d}\Omega \tag{60}$$

$$\mathbf{C}\mathbf{q} = \iint_{\Omega} \mathbf{N}^T\boldsymbol{\varepsilon}\, \mathrm{d}\Omega. \tag{61}$$

We therefore express the strain field in terms of the nodal displacement as shown below:

$$\boldsymbol{\varepsilon} = \mathbf{P}\boldsymbol{\alpha} = \mathbf{P}\mathbf{A}^{-1}\mathbf{C}\mathbf{q} = \mathbf{B}\mathbf{q}. \tag{62}$$

In most cases, it is convenient to take $\mathbf{N} = \mathbf{P}$ in order to obtain a symmetric stiffness matrix. We set $\mathbf{N} = \mathbf{P}$ in this work. The matrix $\mathbf{A}$ in eqn (60) can be easily evaluated. In quasi-conforming elements, the most work is involved in evaluating the matrix $\mathbf{C}$.

Based on the bending strain field, similar to that given by eqn (56), Shi (1980) developed a quadrilateral C^1 element for plate bending problems that gave excellent results. Let us now consider the transverse strain ε_s, which is the most difficult part in the C^0 element, as an example to illustrate the basic concepts of the quasi-conforming technique.

For conciseness, in the following we will use u, v, w, and ϕ_x to imply $\bar{u}$, $\bar{v}$, $\bar{w}$, and ϕ'_x, respectively, and evaluate ε_s in a rectangular element. A typical finite element is illustrated in Fig. 1. The matrices $\mathbf{A}$ and $\mathbf{C}$ obtained for a rectangular element can be transformed to the arbitrary quadrilateral element in the customary approach used for isoparametric elements (see Zienkiewicz, 1977; Chen and Cheung, 1987). Substituting for $\mathbf{P}_s$ from eqn (58) into eqns (60) and (61) and making use of $\mathbf{N}_s = \mathbf{P}_s$, we obtain

$$\mathbf{A}_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \iint_{\Omega} \mathrm{d}\Omega = \Omega \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{63}$$

$$\mathbf{C}_s\mathbf{q} = \iint_{\Omega} \left\{ \begin{matrix} \dfrac{\partial w}{\partial x} - \phi_x \\ \dfrac{\partial w}{\partial y} - \phi_y - \dfrac{v}{R} \end{matrix} \right\} \mathrm{d}\Omega = \oint \left\{ \begin{matrix} w n_x \, \mathrm{d}s \\ w n_y \, \mathrm{d}s \end{matrix} \right\} - \iint_{\Omega} \left\{ \begin{matrix} \phi_x \\ \phi_y + \dfrac{v}{R} \end{matrix} \right\} \mathrm{d}\Omega \tag{64}$$

where Ω is the element area, n_x and n_y are the direction cosines along the element boundaries, and $\mathrm{d}s$ is the differential arc-length along the element boundaries. In order to evaluate $\mathbf{C}_s$, we need to construct the displacement w along the element boundaries, as well as the rotations ϕ_x, ϕ_y and the displacement v over the element. Since the Kirchhoff–Love assumption has to be satisfied for thin shells, consequently, the interpolation for w should be related to the nodal rotation values ϕ_x and ϕ_y. This results in the following relations at node (i) for the case of thin shells:

$$\left.\frac{\partial w}{\partial x}\right|_i = \phi_{xi} \tag{65}$$

$$\left.\frac{\partial w}{\partial y}\right|_i = \phi_{yi} + \frac{v_i}{R} \tag{66}$$

The dependent displacement w and rotation ϕ for a straight beam of length l are given by Hu (1981) as follows:

$$w = \frac{1}{2}\left[1 - \xi + \frac{\lambda}{2}(\xi^3 - \xi)\right]w_i + \frac{1}{4}[1 - \xi^2 + \lambda(\xi^3 - \xi)]\frac{l}{2}\phi_i + \frac{1}{2}\left[1 + \xi - \frac{\lambda}{2}(\xi^3 - \xi)\right]w_j + \tfrac{1}{4}[-1 + \xi^2 + \lambda(\xi^3 - \xi)]\frac{l}{2}\phi_j \tag{67}$$

$$\phi = -\frac{3}{2l}\lambda[1 - \xi^2]w_i + \tfrac{1}{4}[2 - 2\xi - 3\lambda(1 - \xi^2)]\phi_i + \frac{3}{2l}\lambda[1 - \xi^2]w_j + \tfrac{1}{4}[2 + 2\xi - 3\lambda(1 - \xi^2)]\phi_j \tag{68}$$

where

$$\xi = 2x/l \quad -1 \leqslant \xi \leqslant 1 \tag{69}$$

and

$$\lambda = \frac{1}{\left(1 + 12\dfrac{D}{Tl^2}\right)}. \tag{70}$$

In eqn (70), D/Tl^2 is the parameter of the shear deformation effect. It can be seen that $\lambda \to 1$ as $(h/l)^2 \to 0$, and w in eqn (67) reduces to a Hermite function. For a two-dimensional problem, we let L_x be the effective length in the x-direction, and L_y the corresponding effective length in the y-direction. The two-dimensional expressions equivalent to eqn (70) become

$$\lambda_x = \frac{1}{\left(1 + 12\dfrac{D}{TL_x^2}\right)} \tag{71a}$$

and

$$\lambda_y = \frac{1}{\left(1 + 12\dfrac{D}{TL_y^2}\right)}. \tag{71b}$$

In Edwards and Webster's (1976) hybrid stress cylindrical shell element, the explicit rigid body motion is imposed on the edge displacement interpolations which results in a very complicated displacement function and more computing work. Here, we merely use the displacement interpolations w, ϕ, u, and v for a straight beam, while the rigid body motion of a curved element is not considered at all. It will be shown these simple displacement interpolations in QCT can give very good results too.

The interpolations over the element for both ϕ_x and $(\phi_y + v/R)$ are required in eqn (64). However, in the area integrations of expression (64), the integration $\int\int_\Omega \phi_x \, d\Omega$ is obtained from $\int_a \phi_x \, dx$ and $\int_b \phi_x \, dy$ rather than computing it by an explicit interpolation of ϕ_x over the element.

Making use of the following expressions

$$\int_l w \, ds = \frac{l}{2}(w_i + w_j) + \frac{l^2}{12}(\phi_i - \phi_j) \tag{72}$$

$$\int_l \phi \, ds = \lambda(-w_i + w_j) + \frac{1-\lambda}{2} l(\phi_i + \phi_j) \tag{73}$$

and recalling that $\phi_y = \partial w/\partial y - \gamma_y - v/R$, we obtain

$$\oint w n_x \, ds - \int\int_\Omega \phi_x \, dx \, dy = \frac{b}{2}\Bigg\{(1-\lambda_x)[-w_1 - w_2 + w_3 + w_4] - \frac{(1-\lambda_x)}{2} a[\phi_{x1} + \phi_{x2} + \phi_{x3} + \phi_{x4}]$$
$$+ \frac{(1-\lambda_y)}{6} b[-\phi_{y1} + \phi_{y2} - \phi_{y3} + \phi_{y4}] + \frac{(1-\lambda_y)}{6R}[v_1 - v_2 + v_3 - v_4]\Bigg\}. \tag{74}$$

Let $\mathbf{q}$ be the nodal displacement vector expressed as follows:

$$\mathbf{q} = \{u_1, v_1, w_1, \phi_{x1}, \phi_{y1}, u_2, v_2, w_2, \phi_{x2}, \phi_{y2}, u_3, v_3, w_3, \phi_{x3}, \phi_{y3}, u_4, v_4, w_4, \phi_{x4}, \phi_{y4}\}^T. \tag{75}$$

Equation (64) now gives

$$\mathbf{C}_s = \begin{bmatrix} 0 & C_{12} & -C_{13} & -C_{14} & -C_{15} & 0 & -C_{12} & -C_{13} & -C_{14} & C_{15} \\ 0 & C_{22} & -C_{23} & -C_{24} & -C_{25} & 0 & C_{22} & C_{23} & C_{24} & -C_{25} \end{bmatrix.$$
$$\begin{matrix} 0 & C_{12} & C_{13} & -C_{14} & -C_{15} & 0 & -C_{12} & C_{13} & -C_{14} & C_{15} \\ 0 & C_{22} & C_{23} & -C_{24} & -C_{25} & 0 & C_{22} & -C_{23} & C_{24} & -C_{25} \end{bmatrix} \tag{76}$$

where

$$C_{12} = -\frac{1-\lambda_y}{12R} b^2, \quad C_{13} = \frac{1-\lambda_x}{2} b, \quad C_{14} = \frac{1-\lambda_x}{4} ab, \quad C_{15} = \frac{1-\lambda_y}{12} b^2$$
$$C_{22} = -\frac{1-\lambda_y}{4R} ab, \quad C_{23} = \frac{1-\lambda_y}{2} a, \quad C_{24} = \frac{1-\lambda_x}{12} a^2, \quad C_{25} = \frac{1-\lambda_y}{4} ab. \tag{77}$$

It is easy to verify that $\varepsilon_s = 1/\Omega \mathbf{C}_s \mathbf{q} \rightarrow 0$ for thin shells in which $\lambda_y \rightarrow 1$ and $\lambda_x \rightarrow 1$ as $(h/L_x)^2 \rightarrow 0$ and $(h/L_y)^2 \rightarrow 0$.

$\mathbf{C}_m$ for ε_m and $\mathbf{C}_b$ for ε_b can be obtained in a similar way. The explicit forms of $\mathbf{A}_m$, $\mathbf{C}_m$, $\mathbf{A}_b$ and $\mathbf{C}_b$ are given in the Appendix. We finally have:

$$\varepsilon_b = \mathbf{P}_b \mathbf{A}_b^{-1} \mathbf{C}_b \mathbf{q} = \mathbf{B}_b \mathbf{q} \tag{78}$$

$$\varepsilon_m = \mathbf{P}_m \mathbf{A}_m^{-1} \mathbf{C}_m \mathbf{q} = \mathbf{B}_m \mathbf{q} \tag{79}$$

$$\varepsilon_s = \frac{1}{\Omega} \mathbf{C}_s \mathbf{q} = \mathbf{B}_s \mathbf{q}. \tag{80}$$

Substituting eqns (78)–(80) into eqns (50)–(53), we obtain

$$\mathbf{K}_b = \mathbf{C}_b^T \mathbf{A}_b^{-T} \int\int_\Omega \mathbf{P}_b^T \bar{\mathbf{D}} \mathbf{P}_b \, d\Omega \mathbf{A}_b^{-1} \mathbf{C}_b \tag{81}$$

$$\mathbf{K}_m = \mathbf{C}_m^T \mathbf{A}_m^{-T} \int\int_\Omega \mathbf{P}_m^T \bar{\mathbf{S}} \mathbf{P}_m \, d\Omega \mathbf{A}_m^{-1} \mathbf{C}_m \tag{82}$$

$$\mathbf{K}_s = \mathbf{C}_s^T (\bar{\mathbf{T}}/\Omega) \mathbf{C}_s \tag{83}$$

$$\mathbf{K}_{bm} = \mathbf{C}_b^T \mathbf{A}_b^{-T} \int\int_\Omega \mathbf{P}_b^T \mathbf{F} \mathbf{P}_m \, d\Omega \mathbf{A}_m^{-1} \mathbf{C}_m. \tag{84}$$

The external load vector can be evaluated in the usual way. However, the additional load resulting from the radial stress effect as given by eqn (54) and the equivalent distributed moments exerted by distributed load on the surfaces of the shell as shown by Voyiadjis and Shi (1990) should be incorporated for thick shells.

NUMERICAL EXAMPLES

Due to the initial curvature effect, the behavior of curved shell elements is totally different from that of flat plate elements. Many curved shell elements may give quite good results when used in the analysis of shallow shells. On the other hand, they give a poor performance when used in the analysis of deep shells. In the case of moderately thick analysis of shells, we note that C^1 and C^0 shell elements converge satisfactorily. However, they cannot converge to the correct solution for the case of thin shell analysis. Furthermore, some C^0 elements exhibit shear locking. Ashwell and Sabir (1972) pointed out that deep and thin shells are more testing than shallow and moderately thick shells. Therefore, a deep pinched cylindrical shell shown in Fig. 2 is analyzed for two different thicknesses to test the behavior of the C^0 assumed strain element given in the previous section. Only one octant of the shell is considered because of the symmetry.

In the first example, $p = 100$ lbf, $h = 0.094$ in. which results in $R/h = 53$, a moderately thick shell. The deflections at the load point obtained by different elements are listed in Table 1 in which Park and Stanley's results (1986) are those given by their 4-ANS C^0 element.

The second example concerns a thin shell in which $R/h = 320$ ($h = 0.01548$ in.) and

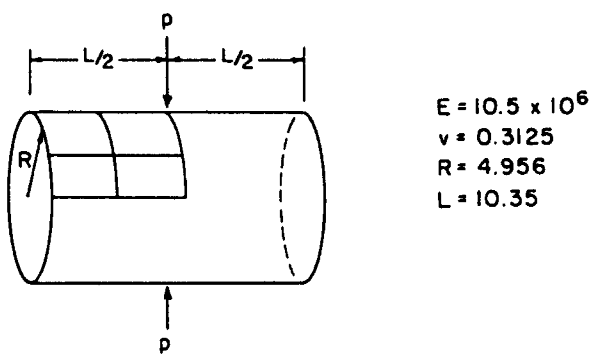

Fig. 2. Pinched cylindrical shell.

Table 1. Deflection for moderately thick pinched cylinder

Mesh	Present	Ashwell and Sabir (1972)	Park and Stanley (1986)	Mesh	Cantin and Clough (1968)
2×2	0.904	0.1103	0.0703	(1×5)	0.0769
4×4	0.1068	0.1129	0.1002	(2×9)	0.1073

Table 2. Deflection for thin cylindrical shell

Mesh	Present	Cantin and Clough (1968)	Ashwell and Sabir (1972)	Sabir and Lock (1972)
1 × 1	0.01091	0.00001	0.02301	0.00001
1 × 2	0.02240			
1 × 4	0.02408	0.00074	0.02403	0.00063
8 × 8		0.00708	0.02431	0.00706

$p = 0.1$ lbf. Ashwell and Sabir (1972) stated that the analytical solution of this problem is 0.02439 in. The deflections given by different researchers are tabulated in Table 2. As mentioned earlier, even though some shell elements can give quite good results in the analysis of moderately thick shells, they may exhibit very poor performance in the case of thin shell analysis. Furthermore, they may not converge to the correct solution even for a very fine mesh. The two tables show that the present C^0 quasi-conforming element converges very fast and gives very good results for both thick and thin shell analysis.

CLOSURE

By the simple modification of the constitutive equations, the refined two-dimensional shell theory proposed by the authors can be easily applied to the finite element analysis. Unlike most shell theories, the strain energy density resulting from the proposed refined shell theory is a coupled strain energy (between bending strains and stretch strains) and is an explicit function of the distributed load. Therefore, the coupled strain energy density presented here takes account for not only the transverse shear strains, but also the initial curvature effect as well as the contribution of the radial stresses in the shells.

A simple and efficient C^0 quadrilateral shell element is developed here based on the coupled strain energy density and the quasi-conforming element technique. The stiffness matrix presented here is given explicitly. This quasi-conforming C^0 shell element is valid for both thick and thin shell analysis. All the deficiencies encountered in the construction of curved C^0 elements can be overcome very easily through the quasi-conforming element technique. In quasi-conforming elements, the rigid body motion can be guaranteed automatically by the assumed strain fields for both flat and curved elements; the spurious zero energy modes can be prevented by the proper strain fields for the given element nodal variables without differences for both flat and curved elements too; the shear locking can be avoided by the dependent displacement and rotation interpolations for a simple Timoshenko beam which satisfies the Kirchhoff–love hypothesis in the case of the thin plates and shells. Therefore, the quasi-conforming element technique is a natural and powerful approach in the formulation of various types of finite elements. The numerical examples solved here show the quasi-conforming C^0 shell element gives good results for both thick and thin shells. This element is extremely efficient for nonlinear analysis of shells since there is no numerical integration used in the formulation of the stiffness matrix.

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APPENDIX

The explicit forms of $\mathbf{A}_m$, $\mathbf{C}_m$, $\mathbf{A}_m$ *and* $\mathbf{C}_b$

$$\mathbf{A}_m = ab \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2/12 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b^2/12 \end{bmatrix} \tag{A1}$$

$$\mathbf{C}_m = \left[\begin{array}{cccccccccc} -m_1 & 0 & 0 & 0 & 0 & -m_1 & 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 & 0 & -m_2 & 0 & 0 & 0 & 0 \\ 0 & -m_{32} & m_{33} & m_{34} & m_{35} & 0 & m_{32} & m_{33} & m_{34} & -m_{35} \\ 0 & m_{42} & -m_{43} & m_{44} & 0 & 0 & -m_{42} & -m_{43} & m_{44} & 0 \\ -m_5 & -m_1 & 0 & 0 & 0 & m_5 & -m_1 & 0 & 0 & 0 \end{array} \right.$$

$$\left. \begin{array}{cccccccccc} m_1 & 0 & 0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 \\ m_2 & 0 & 0 & 0 & 0 & -m_2 & 0 & 0 & 0 & 0 \\ 0 & m_{32} & m_{33} & -m_{34} & m_{35} & 0 & -m_{32} & m_{33} & -m_{34} & m_{35} \\ 0 & m_{42} & m_{43} & m_{44} & 0 & 0 & -m_{42} & m_{43} & m_{44} & 0 \\ m_5 & m_1 & 0 & 0 & 0 & -m_5 & m_1 & 0 & 0 & 0 \end{array} \right], \tag{A2}$$

in which

$$m_1 = \frac{b}{2},\quad m_2 = \frac{b^2}{12},\quad m_{32} = \frac{a}{2} - \frac{a}{24}\left(\frac{b}{R}\right)^2,\quad m_{33} = \frac{ab}{4R},\quad m_{34} = \frac{a^2b}{24R},$$

$$m_{35} = \frac{ab^2}{24R},\quad m_{42} = \frac{a^2}{12},\quad m_{43} = \frac{a^2b}{24R},\quad m_{44} = \frac{ab^3}{240R},\quad m_5 = \frac{a}{2} \tag{A3}$$

$$\mathbf{A}_h = \begin{bmatrix} \mathbf{A}_{11} & 0 & 0 \\ 0 & \mathbf{A}_{11} & 0 \\ 0 & 0 & \mathbf{A}_{22} \end{bmatrix}, \tag{A4}$$

where

$$\mathbf{A}_{11} = ab \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{12} & 0 & 0 \\ 0 & 0 & \frac{b^2}{12} & 0 \\ 0 & 0 & 0 & \frac{a^2b^2}{144} \end{bmatrix} \tag{A5}$$

$$\mathbf{A}_{22} = ab \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{a^2}{12} & 0 \\ 0 & 0 & \frac{b^2}{12} \end{bmatrix} \tag{A6}$$

$$\mathbf{C}_h = ab \left[\begin{array}{cccccccccc}
0 & 0 & 0 & -b_1 & 0 & 0 & 0 & 0 & -b_1 & 0 \\
0 & b_{22} & b_{23} & b_{24} & b_{25} & 0 & -b_{22} & b_{23} & b_{24} & b_{25} \\
0 & 0 & 0 & b_{14} & 0 & 0 & 0 & 0 & -b_{34} & 0 \\
0 & -b_{42} & -b_{43} & -b_{44} & -b_{45} & 0 & -b_{42} & b_{43} & b_{44} & -b_{45} \\
0 & 0 & 0 & 0 & -b_{55} & 0 & 0 & 0 & 0 & b_{55} \\
0 & 0 & 0 & 0 & b_{65} & 0 & 0 & 0 & 0 & -b_{65} \\
0 & b_{72} & b_{73} & b_{74} & b_{75} & 0 & b_{72} & -b_{73} & -b_{74} & b_{75} \\
0 & -b_{82} & -b_{83} & -b_{84} & -b_{85} & 0 & -b_{82} & b_{83} & b_{84} & -b_{85} \\
-b_{91} & b_{92} & b_{93} & -b_{94} & -b_{95} & b_{91} & b_{92} & -b_{93} & b_{94} & -b_{95} \\
b_{01} & b_{02} & b_{03} & b_{04} & b_{05} & -b_{01} & b_{02} & -b_{03} & -b_{04} & b_{05} \\
0 & -b_{12} & b_{13} & b_{14} & b_{15} & 0 & b_{12} & b_{13} & b_{14} & -b_{15}
\end{array}\right.$$

$$\left.\begin{array}{cccccccccc}
0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & b_1 & 0 \\
0 & -b_{22} & b_{23} & b_{24} & b_{25} & 0 & b_{22} & -b_{23} & b_{24} & b_{25} \\
0 & 0 & 0 & b_{34} & 0 & 0 & 0 & 0 & -b_{14} & 0 \\
0 & b_{42} & -b_{43} & b_{44} & b_{45} & 0 & b_{42} & b_{43} & -b_{44} & b_{45} \\
0 & 0 & 0 & 0 & b_{55} & 0 & 0 & 0 & 0 & -b_{55} \\
0 & 0 & 0 & 0 & b_{65} & 0 & 0 & 0 & 0 & -b_{65} \\
0 & b_{72} & -b_{73} & b_{74} & b_{75} & 0 & b_{72} & b_{73} & -b_{74} & b_{75} \\
0 & b_{82} & -b_{83} & b_{84} & b_{85} & 0 & b_{82} & b_{83} & -b_{84} & b_{85} \\
b_{91} & -b_{92} & b_{93} & b_{94} & b_{95} & -b_{91} & -b_{92} & -b_{93} & -b_{94} & b_{95} \\
b_{01} & b_{02} & -b_{03} & b_{04} & b_{05} & -b_{01} & b_{02} & b_{03} & -b_{04} & b_{05} \\
0 & -b_{12} & -b_{13} & b_{14} & b_{15} & 0 & b_{12} & -b_{13} & b_{14} & -b_{15}
\end{array}\right] \tag{A7}$$

in which

$$b_1 = \frac{1}{2a},\quad b_{22} = \frac{\lambda_y b}{12aR},\quad b_{23} = \frac{\lambda_x}{2a},\quad b_{24} = \frac{\lambda_x}{4},\quad b_{25} = \frac{\lambda_y b}{12a},$$

$$b_{34} = \frac{b}{12a},\quad b_{42} = \frac{b^2}{120Ra},\quad b_{43} = \frac{b}{10a},\quad b_{44} = \frac{b}{24},\quad b_{45} = \frac{b^2}{120a},$$

$$b_{55} = \frac{1}{2b},\quad b_{65} = \frac{a}{12b},\quad b_{72} = \frac{\lambda_y}{4R},\quad b_{73} = \frac{\lambda_y}{2b},\quad b_{74} = \frac{\lambda_x a}{12b},$$

$$b_{75} = \frac{\lambda_y}{4},\quad b_{82} = \frac{\lambda_y a}{24R},\quad b_{83} = \frac{a}{10b},\quad b_{84} = \frac{a2}{120b},\quad b_{85} = \frac{a}{24},$$

$$b_{91} = \frac{1}{4Rb},\quad b_{92} = \frac{1}{4Ra},\quad b_{93} = \frac{\lambda_x}{ab},\quad b_{94} = \frac{1-\lambda_x}{2b},\quad b_{95} = \frac{1}{2a},$$

$$b_{01} = \frac{a}{24Rb},\quad b_{02} = \frac{\lambda_y}{4R},\quad b_{03} = \frac{\lambda_y}{2b},\quad b_{04} = \frac{1+\lambda_x}{12}\frac{a}{b},\quad b_{05} = \frac{\lambda_y}{4},$$

$$b_{12} = \frac{1-2\lambda_y}{24R}\frac{b}{a},\quad b_{13} = \frac{\lambda_x}{2a},\quad b_{14} = \frac{\lambda_x}{4},\quad b_{15} = \frac{1+\lambda_y}{12}\frac{b}{a}. \tag{A8}$$